Thermodynamics of the localized D2–D6 System

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Abstract

An exact fully-localized extremal supergravity solution for N_2 D2 branes and N_6 D6 branes, which is dual to 3-dimensional supersymmetric $SU(N_2)$ gauge theory with N_6 fundamentals, was found by Cherkis and Hashimoto. In order to consider the thermal properties of the gauge theory we present the non-extremal extension of this solution to first order in an expansion near the core of the D6 branes. We compute the Hawking temperature and the black brane horizon area/entropy. The leading order entropy, which is proportional to $N_2^{3/2} N_6^{1/2} T_H^2$, is not corrected to first order in the expansion. This result is consistent with the analogous weak-coupling result at the correspondence point $N_2 \sim N_6$.

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1   Introduction

An important issue for the gauge/gravity correspondence is how to interpolate between strong gauge coupling (where the supergravity approximation is valid) and the weakly-coupled regime of the gauge theory. In this context there have been several different approaches to the search for string duals to large N gauge theories at weak coupling. The study of the thermodynamics of various systems is one tool in analyzing this question [1, 2]. Typically the thermal behavior of large N gauge theories exhibits a deconfining transition at sufficiently high temperature, which is dual to a phase-transition from a thermal state to a black hole in the string theory [3]. Such transitions can occur at both weak and strong coupling, so that study of the thermal behavior of both gauge theories and their dual string theories is well suited for investigation of aspects of the gauge/gravity duality.

One approach is to consider large N gauge theories on a compact space, as this provides an additional parameter $R\Lambda$, which may be tuned to weak coupling, where R is the size of the compact manifold, and Λ the dynamical scale of the gauge theory. References [4] and [5] have provided a general framework for this discussion on the gauge side. In that context, one can consider SU(N) gauge theories with N_f matter multiplets in the fundamental representation of the gauge group, with N_f/N finite in the large N limit. It has been shown that in the weak-coupling limit, this class of gauge theories on $S^{d-1}\times$ *time* has two phases [6, 7], separated by a third-order phase transition at temperature T_c. The free energy in the low-temperature phase behaves as

$$\frac{F}{T} \sim N_f^2\, f_{\text{low}}(T)\,, \qquad T \leq T_c\,, \tag{1.1}$$

whereas in the high-temperature phase,

$$\frac{F}{T} \sim N^2\, f_{\text{high}}\left(N_f/N\,,\ T\right)\,, \qquad T \geq T_c\,, \tag{1.2}$$

where $f_{\text{low}}(T_c) = (N/N_f)^2 f_{\text{high}}(N_f/N\,,\ T_c)$. The high-temperature limit of (1.2) becomes [6]

$$F \sim N^2 T^d \tilde{f}(N_f/N)\,. \tag{1.3}$$

This can be interpreted as the behavior of glueballs and (color-singlet) mesons at low temperature, with a deconfining transition to a phase of gluons and fundamental (and anti-fundamental) matter at high temperature. It was speculated [6] that the low-temperature phase was dual to a thermal string state, with a high-temperature transition to a black hole.

In this paper, we study the thermal behavior of a string theory dual to a gauge theory of this type. One might be tempted to consider the large N limit of type IIB theory with N D3 branes and N_f D7 branes, with N_f/N finite, but the conical singularities associated with D7 branes limits their number. A cleaner example, which can be used to study $d=3$ gauge theory, involves type IIA theory with N_2 D2 branes and N_6 D6 branes, where there is no restriction on the number of D6 branes. A fully localized D2–D6 solution is required for the dual to the SU(N_2) gauge theory with N_6 fundamentals. In ref. [8] the exact extremal (*i.e.*, zero-temperature) metric for localized D2 and D6 branes was obtained. Previously, the approximate (extremal) metric for a localized D2–D6 system valid near the core of the D6 branes was given in [9]. This metric was also considered in [10] in the context of describing a gravity dual of mesons for this gauge theory.

In order to discuss the thermal properties of the gauge/string duality, one needs the non-extremal analog of the extremal solution presented in [8]. We have not been able to obtain an exact non-extremal solution of the localized D2-D6 system, so we consider a systematic expansion near the core of the D6 branes. The non-extremal metric in the near-core region (corresponding to the IR fixed-point of the gauge theory) was obtained in [11], and corresponds to the leading term in our expansion. In section 3, we obtain the first-order correction to this non-extremal metric. The correction to the metric involves the beginning of the flow away from the IR fixed point.

In section 4, we examine the thermodynamics implied by our solution in the decoupling limit, which uncouples the $SU(N_2)$ gauge theory from the bulk, and leaves a $SU(N_6)$ global flavor symmetry from the D6 branes. We compute the Hawking temperature T_H and the area of the black-brane horizon, which is proportional to the entropy, for our solution. We find the entropy as a function of temperature

$$ S = \frac{8\pi^2}{27} \sqrt{2N_2^3 N_6} \, V_2 \, T_H^2 \,, \tag{1.4} $$

valid to first-order in our expansion. As is evident from (1.4), our calculation is that of the high-temperature limit, *i.e.*, of the black-hole thermodynamics in $d=3$.

The validity of the geometrical description requires large $N_2 N_6$. As one varies N_6 relative to N_2, the theory has different phases [11]. For small N_6, the theory is 11-dimensional, with geometry $AdS_4 \times S_7/\mathbb{Z}_{N_6}$. As N_6/N_2 increases, the geometry becomes 10-dimensional, with AdS_4 fibered over a compact X_6. When $N_6 \gg N_2$, the 10-dimensional geometry becomes highly curved, and one passes to a weakly-coupled phase of the gauge theory. This suggests a "correspondence point" between the 10-dimensional sugra regime and perturbative gauge theory [6], estimated to be at

$$ N_2 \sim N_6 \equiv N \,. \tag{1.5} $$

Applying this correspondence to (1.4), one has

$$ S \sim N^2 \, T_H^2 \,, \tag{1.6} $$

appropriate to a black-hole, and in agreement with the high-temperature limit (1.3).

2 Extremal 11-dimensional supergravity solution

A localized D2–D6 brane configuration in type IIA string theory uplifts to an M-theory configuration consisting of M2 branes in a Taub–NUT background. The supergravity solution corresponding to this configuration satisfies the 11-dimensional equations of motion

$$ R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{1}{12}\left(F_{\mu\alpha\beta\gamma}F_\nu{}^{\alpha\beta\gamma} - \tfrac{1}{8}g_{\mu\nu}F_{\alpha\beta\gamma\delta}F^{\alpha\beta\gamma\delta}\right), \tag{2.1} $$

$$ \mathrm{d}\,{}^*F_{[4]} + \tfrac{1}{2}F_{[4]}\wedge F_{[4]} = 0, \tag{2.2} $$

which follow from the bosonic part of the 11-dimensional supergravity action

$$ I = \frac{1}{16\pi G_{11}} \int \mathrm{d}^{11}x \left[\sqrt{-g}\left(R - \tfrac{1}{48}F_{[4]}^2\right) + \tfrac{1}{6}F_{[4]}\wedge F_{[4]}\wedge A_{[3]}\right]. \tag{2.3} $$

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In the absence of M2–branes, $F_{[4]}$ vanishes, and the flat space equations $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0$ have as a solution the ($\mathbb{R}^7 \times$ Taub-NUT) metric, where the Taub-NUT metric may be written as

$$\mathrm{d}s_{TN}^2 = \left(1 + \frac{2mN_6}{r}\right)\left[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right] + \frac{(4m)^2}{(1 + \frac{2mN_6}{r})}\left(\mathrm{d}\psi + \frac{N_6}{2}\cos\theta\,\mathrm{d}\phi\right)^2, \quad (2.4)$$

with $0 \le \theta < \pi$, $0 \le \phi < 2\pi$, and $0 \le \psi < 2\pi$. The radius of the circle of the Taub-NUT metric at $r = \infty$ is $R_\# = 4m$. This solution corresponds to the M-theory lift of a configuration of N_6 coincident D6 branes located at $r = 0$ and spanning $\mathbb{R}^7$.

If M2 branes are present, they act as a source for $F_{[4]}$. The extremal supergravity solution for M2 branes in the ($\mathbb{R}^7 \times$ Taub-NUT) background was derived by Cherkis and Hashimoto in ref. [8]. For N_2 parallel M2-branes spanning the t, x^1, x^2 directions they used the following ansatz

$$\mathrm{d}s_{11}^2 = H^{-2/3}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2) + H^{1/3}(\mathrm{d}y^2 + y^2\mathrm{d}\Omega_3^2 + \mathrm{d}s_{TN}^2), \quad (2.5)$$

$$F_{[4]} = \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}H^{-1}, \quad (2.6)$$

where

$$\mathrm{d}\Omega_3^2 = \mathrm{d}\alpha_1^2 + \sin^2\alpha_1\left(\mathrm{d}\alpha_2^2 + \sin^2\alpha_2\,\mathrm{d}\alpha_3^2\right). \quad (2.7)$$

If the M2 branes are coincident, and lie at $r = y = 0$, the function H only depends on r and y. Then the ansatz (2.6) substituted into eq. (2.2) yields

$$0 = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{012\mu}\right) = \frac{1}{\left(1 + \frac{2mN_6}{r}\right)}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}H(r,y)\right) + \frac{1}{y^3}\frac{\partial}{\partial y}\left(y^3\frac{\partial}{\partial y}H(r,y)\right). \quad (2.8)$$

In ref. [8] this equation is explicitly solved as the Fourier transform of the confluent hypergeometric function.

The aim of this paper is to find a non-extremal generalization of this localized D2–D6 brane solution. In order to do so, the first step is to use the change of variables $r = z^2/8mN_6$ to rewrite the metric (2.5) as

$$\begin{aligned}\mathrm{d}s_{11}^2 &= H^{-2/3}(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2) + H^{1/3}\Bigg\{\mathrm{d}y^2 + y^2\mathrm{d}\Omega_3^2 + \left[1 + \left(\frac{z}{4mN_6}\right)^2\right]\mathrm{d}z^2 \\ &\quad + \left(\frac{z}{2}\right)^2\left[1 + \left(\frac{z}{4mN_6}\right)^2\right](\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \\ &\quad + z^2\left[1 + \left(\frac{z}{4mN_6}\right)^2\right]^{-1}\left(\frac{\mathrm{d}\psi}{N_6} + \frac{1}{2}\cos\theta\,\mathrm{d}\phi\right)^2\Bigg\}.\end{aligned} \quad (2.9)$$

In these variables the equation (2.8) for H takes the form

$$\frac{1}{z^3}\frac{\partial}{\partial z}\left(z^3\frac{\partial}{\partial z}H(z,y)\right) + \frac{1}{y^3}\frac{\partial}{\partial y}\left(y^3\frac{\partial}{\partial y}H(z,y)\right) = -\left(\frac{z}{4mN_6}\right)^2\frac{1}{y^3}\frac{\partial}{\partial y}\left(y^3\frac{\partial}{\partial y}H(z,y)\right). \quad (2.10)$$

One can solve this equation for $H(z,y)$ order-by-order in an expansion in $1/m$, with the leading order given by the solution found in refs. [9, 11] for a localized D2-D6 system near the core of the D6 branes.

There is a further change of variables that can be made to simplify the computations. We seek variables (R, β) such that

$$dy^2 + \left[1 + \left(\frac{z}{4mN_6} \right)^2 \right] dz^2 = dR^2 + R^2 \, d\beta^2 \,. \tag{2.11}$$

The change of variables between the set (y, z) and the set (R, β) is given by the relations

$$y = R \cos \beta, \qquad f(z) = R \sin \beta \,, \tag{2.12}$$

where the function $f(z)$ is the solution of the differential equation

$$df = \sqrt{1 + \left(\frac{z}{4mN_6} \right)^2} \, dz \,, \tag{2.13}$$

given by

$$\begin{aligned} f(z) &= \frac{z}{2} \sqrt{1 + \left(\frac{z}{4mN_6} \right)^2} + 2mN_6 \operatorname{arcsinh} \left(\frac{z}{4mN_6} \right) \\ &= z \left[1 + \frac{1}{6} \left(\frac{z}{4mN_6} \right)^2 - \frac{1}{40} \left(\frac{z}{4mN_6} \right)^4 + \dots \right] . \end{aligned} \tag{2.14}$$

Equation (2.14) may be inverted in an expansion in $1/m$ to give

$$z = R \sin \beta \left[1 - \frac{B^2}{6m^2} + \frac{13B^4}{120m^4} + \dots \right], \qquad B = \frac{R \sin \beta}{4N_6} \,. \tag{2.15}$$

Now, using (2.11) and (2.15), one may rewrite the extremal metric (2.9) in an expansion in $1/m$

$$\begin{aligned} ds_{11}^2 &= H^{-2/3}(-dt^2 + dx_1^2 + dx_2^2) + H^{1/3} \Bigg[dR^2 + R^2 d\beta^2 + (R \cos \beta)^2 d\Omega_3^2 \\ &+ \left(\frac{R \sin \beta}{2} \right)^2 \left(1 + \frac{2B^2}{3m^2} - \frac{19B^4}{45m^4} \right) (d\theta^2 + \sin^2 \theta \, d\phi^2) \\ &+ (R \sin \beta)^2 \left(1 - \frac{4B^2}{3m^2} + \frac{86B^4}{45m^4} \right) \left(\frac{d\psi}{N_6} + \tfrac{1}{2} \cos \theta \, d\phi \right)^2 \Bigg] + \mathcal{O} \left(\frac{1}{m^6} \right) . \end{aligned} \tag{2.16}$$

Higher order terms can be generated easily if needed. In the limit $m \to \infty$, the Taub-NUT space reduces to the orbifold $\mathbb{R}^4 / \mathbb{Z}_{N_6}$, and the metric (2.16) reduces to that obtained in [9, 11].

The solution for H may be written in an expansion in $1/m$ in the (R, β) variables:

$$H(R, \beta) = 1 + Q \sum_{n=0}^{\infty} \frac{h_n(\beta)}{R^{6-2n}(4mN_6)^{2n}} \,. \tag{2.17}$$

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Up to order $1/m^4$, we find

$$H = 1 + Q\left[\frac{1}{R^6} + \frac{1}{15(4mN_6)^4 R^2}\left(1 + \sin^2\beta + \sin^4\beta\right)\right] + \mathcal{O}\left(\frac{1}{m^6}\right), \qquad (2.18)$$

where $Q = 32\pi^2 N_2 N_6 \ell_p^6$ [9]. Note that the first correction to H occurs at $\mathcal{O}(1/m^4)$ rather than $\mathcal{O}(1/m^2)$. This will be useful for finding the non-extremal version of the metric (2.16) in the next section.

3   Non-extremal 11-dimensional supergravity solution

We have not been able to find an exact non-extremal solution generalizing the extremal solution of ref. [8], so we turn instead to find an approximate non-extremal solution, based on the $1/m$ expansion developed in the last section. We make the following ansatz for the non-extremal metric and antisymmetric tensor field

$$
\begin{aligned}
\mathrm{d}s_{11}^2 &= H^{-2/3}(-f_1 \mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2) + H^{1/3}\Big[f_1^{-1}\mathrm{d}R^2 + R^2\mathrm{d}\beta^2 + (R\cos\beta)^2\mathrm{d}\Omega_3^2 \\
&+ f_2\left(\frac{R\sin\beta}{2}\right)^2\left(1 + \frac{2B^2}{3m^2}\right)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \\
&+ f_2^{-2}(R\sin\beta)^2\left(1 - \frac{4B^2}{3m^2}\right)\left(\frac{\mathrm{d}\psi}{N_6} + \tfrac{1}{2}\cos\theta\,\mathrm{d}\phi\right)^2\Big] + \mathcal{O}\left(\frac{1}{m^4}\right), \\
F_{[4]} &= \mathrm{d}t \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\hat{H}^{-1}.
\end{aligned}
$$

$$\text{(3.1)}$$
$$\text{(3.2)}$$

Note that the function $\hat{H}$ in $F_{[4]}$ is distinct from the function H in the metric.

First we consider the $m \to \infty$ limit, where the ($\mathbb{R}^7 \times$ Taub-NUT) background reduces to $\mathbb{R}^7$ times the $\mathbb{Z}_{N_6}$ orbifold of $\mathbb{R}^4$. In that case, the non-extremal solution is given by [11, 12]

$$
\begin{aligned}
f_1 &= 1 - \left(\frac{R_H}{R}\right)^6, \\
f_2 &= 1, \\
H &= 1 + \frac{Q}{R^6}, \qquad\qquad\qquad \text{for}\quad m = \infty. \\
\hat{H} &= \left[1 - \frac{Q\sqrt{1 + R_H^6/Q}}{Q + R^6}\right]^{-1},
\end{aligned}
$$

$$\text{(3.3)}$$

The Hawking temperature associated with this black brane metric is

$$T_H = \frac{3}{2\pi R_H \sqrt{H(R_H)}}, \qquad (3.4)$$

found in the usual way by requiring the absence of a conical singularity at the horizon $R = R_H$ of the Euclidean continuation.

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Let us now consider the non-extremal solution away from $m = \infty$. As seen from eq. (2.18), H receives no corrections at $\mathcal{O}(1/m^2)$. Using this we make the ansatz that, at $\mathcal{O}(1/m^2)$, f_1 and $\hat{H}$ are also unchanged from (3.3), but that f_2 takes the form

$$f_2 = 1 + \left(\frac{R}{4mN_6}\right)^2 f_R(R)f_\beta(\beta) + \mathcal{O}\left(\frac{1}{m^4}\right).$$
(3.5)

The Einstein equations (2.1) can be written as

$$R_{\mu\nu} + g_{\mu\nu}R = \tfrac{1}{12}F_{\mu\alpha\beta\gamma}F_\nu{}^{\alpha\beta\gamma}.$$
(3.6)

Since, to the order we are working, $\hat{H}$ only depends on R, the r. h. s. only receives contributions from t, x^1, x^2, and R components. One can check that the l. h. s. of (3.6) vanishes (to $\mathcal{O}(1/m^2)$) for all other components except the $\theta\theta$, $\phi\phi$, $\phi\psi$, and $\psi\psi$ components. Requiring that these components also vanish implies that $f_\beta(\beta) = \sin^2\beta$ and that $f_R(R)$ satisfies

$$3\,R^2\left(R^6 - R_H{}^6\right)f_R''(R) \;+\; 3R\left(11\,R^6 - 5\,R_H{}^6\right)f_R'(R) \;-\; 12\,R_H{}^6\,f_R(R) = 8R_H{}^6.$$
(3.7)

The solution to this equation may be written in terms of the hypergeometric function:

$$f_R(R) = \frac{2}{3}\left[\, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6) - 1\right]$$
(3.8)

so therefore

$$f_2(R, \beta) = 1 + \frac{2}{3}\left(\frac{R}{4mN_6}\right)^2\left[\, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6) - 1\right]\sin^2\beta + \mathcal{O}\left(\frac{1}{m^4}\right).$$
(3.9)

With this form for $f_2(R, \beta)$, all the Einstein equations, as well as the antisymmetric tensor field equation, are satisfied at order $1/m^2$. Hence we have obtained the approximate non-extremal solution

$$\begin{aligned}
ds_{11}^2 &= H^{-2/3}(-f_1 dt^2 + dx_1^2 + dx_2^2) + H^{1/3}\Bigg\{ f_1^{-1}dR^2 + R^2 d\beta^2 + (R\cos\beta)^2 d\Omega_3^2 \\[2mm]
&\quad + \left(\frac{R\sin\beta}{2}\right)^2\left[1 + \frac{2B^2}{3m^2}\, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6)\right](d\theta^2 + \sin^2\theta\,d\phi^2) \\[2mm]
&\quad + (R\sin\beta)^2\left[1 - \frac{4B^2}{3m^2}\, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6)\right]\left(\frac{d\psi}{N_6} + \tfrac{1}{2}\cos\theta\,d\phi\right)^2\Bigg\} + \mathcal{O}\left(\frac{1}{m^4}\right)
\end{aligned}$$

$$F_{[4]} = dt \wedge dx_1 \wedge dx_2 \wedge d\hat{H}^{-1}$$
(3.10)

with

$$\begin{aligned}
f_1 &= 1 - \left(\frac{R_H}{R}\right)^6 + \mathcal{O}\left(\frac{1}{m^4}\right) \\[2mm]
H &= 1 + \frac{Q}{R^6} + \mathcal{O}\left(\frac{1}{m^4}\right) \\[2mm]
\hat{H} &= \left[1 - \frac{Q\sqrt{1 + R_H^6/Q}}{Q + R^6}\right]^{-1} + \mathcal{O}\left(\frac{1}{m^4}\right)
\end{aligned}$$
(3.11)

The horizon of the black brane approximate solution (3.10) is given by $R = R_H + \mathcal{O}(1/m^4)$.

We observe that the approximate solution (3.10) will be valid as long as the correction term is smaller than the leading term, thus $B/m \ll 1$ or $R \ll 4mN_6/\sin\beta$. However, the hypergeometric function multiplying the $1/m^2$ correction diverges logarithmically at $R = R_H$, so we must also have $R \gg R_H$. This is unfortunate, as we are particularly interested in the behavior of the solution at the horizon. On the other hand, the fact that some components of the metric seem to diverge at the horizon may be an artifact of our $1/m$ expansion, and it is conceivable that f_2, summed to all orders in $1/m$, is perfectly regular.

Formally, the computation of the Hawking temperature of the metric (3.1) is independent of f_2, and is given by eq. (3.4) through order $1/m^2$. Thus, while our approximate metric clearly breaks down at the horizon, there is no evidence of this breakdown in the Hawking temperature.

Furthermore, the area of the horizon of the black brane computed using (3.1),

$$A = \frac{\pi^4 R_H^7}{3N_6}\sqrt{H(R_H)}V_2\,, \qquad V_2 = \int \mathrm{d}x_1 \mathrm{d}x_2 \qquad (3.12)$$

is also formally independent of f_2, and thus naively unaffected by the (apparent) divergence of f_2 at the horizon. By the Bekenstein-Hawking relation, the entropy $S = A/4G_{11}$ is similarly unaffected at order $1/m^2$. Therefore, while our $1/m^2$ approximation shows large corrections to the metric near the black brane horizon (and in fact breaks down there), both the Hawking temperature and the black brane horizon area are, superficially at least, insensitive to the $1/m^2$ corrections and therefore given by their $m \to \infty$ values, without correction.

Higher orders in $1/m$ probably require a further generalization (beyond inclusion of the function f_2) of the non-extremal metric and antisymmetric tensor field ansatz which may (or may not) affect the value of the Hawking temperature and horizon area. However, to $1/m^2$ at least, we see no evidence of any correction to these quantities. One could further speculate that the $m \to \infty$ results for the Hawking temperature and the horizon area/entropy are valid for all m.

4   Decoupling limit

Next, we consider the decoupling limit [13] of our 11-dimensional approximate solution (3.10). For that purpose, we define

$$R^2 = \ell_p^3 U\,, \qquad R_H^2 = \ell_p^3 U_H\,, \qquad (4.1)$$

where ℓ_p is the 11-dimensional Planck length. In the decoupling limit $\ell_p \to 0$, with U fixed, one has

$$H = 1 + \frac{32\pi^2 N_2 N_6}{\ell_p^3 U^3} \longrightarrow \frac{32\pi^2 N_2 N_6}{\ell_p^3 U^3}, \qquad (4.2)$$

yielding the following decoupled metric

$$\ell_p^{-2}\mathrm{d}s_{11}^2 = U^2(32\pi^2 N_2 N_6)^{-2/3}\left(-f_1\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + (32\pi^2 N_2 N_6)^{1/3}\left\{f_1^{-1}\frac{\mathrm{d}U^2}{4U^2} + \mathrm{d}\beta^2\right.$$

$$+ \cos^2\beta \, d\Omega_3^2 + \tfrac{1}{4}\sin^2\beta \left[1 + \frac{2U\sin^2\beta}{3g_{\text{YM}}^2 N_6^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}; U_H^3/U^3)\right] (d\theta^2 + \sin^2\theta \, d\phi^2)$$

$$+ \sin^2\beta \left[1 - \frac{4U\sin^2\beta}{3g_{\text{YM}}^2 N_6^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}; U_H^3/U^3)\right] \left(\frac{d\psi}{N_6} + \tfrac{1}{2}\cos\theta \, d\phi\right)^2\Bigg\} + \mathcal{O}\left(\frac{1}{m^4}\right),$$

(4.3)

where we have used $\ell_p^3 = g_s\ell_s^3$, $4m = g_s\ell_s$, and $g_{\text{YM}}^2 = g_s/\ell_s$ (where g_{YM} is the $3d$ gauge coupling on the D2-brane) to express

$$\left(\frac{B}{m}\right)^2 = \left(\frac{R\sin\beta}{4mN_6}\right)^2 = \frac{U\sin^2\beta}{g_{\text{YM}}^2 N_6^2}.$$

(4.4)

From the decoupled metric (4.3) we can compute the Ricci scalar

$$R_{11} = \left(\frac{27}{4\pi^2 N_2 N_6}\right)^{1/3} \frac{1}{\ell_p^2} + \mathcal{O}\left(\frac{1}{m^4}\right).$$

(4.5)

The validity of the 11-dimensional supergravity solution requires that the Ricci curvature R_{11} be small compared to ℓ_p^{-2}, which is satisfied provided $N_2 N_6 \gg 1$. Moreover, the validity of the $1/m$ expansion employed in this paper requires that the corrections be small, i.e. $B/m \ll 1$, which implies

$$U \ll \frac{g_{\text{YM}}^2 N_6^2}{\sin^2\beta}.$$

(4.6)

Simultaneously, we must impose $U > U_H$ since the $1/m$ expansion appears to break down near the horizon $U \sim U_H$ due to a logarithmic divergence of the hypergeometric function. (But note that the Ricci curvature (4.5) is unaffected by the $1/m^2$ corrections to the metric, and in particular is finite at the horizon.)

As we have noted above, neither the Hawking temperature (3.4) nor the horizon area (3.12) formally have any $\mathcal{O}(1/m^2)$ corrections. If we assume that the divergence at $U = U_H$ is an artifact of the approximation scheme, and that in fact the Hawking temperature and horizon area are unchanged to $\mathcal{O}(1/m^2)$, we can compute their values in the decoupling limit to be

$$T_H = \frac{3U_H}{8\pi^2\sqrt{2N_2 N_6}}$$

(4.7)

and[4]

$$A = \frac{4\pi^5}{3}\sqrt{\frac{2N_2}{N_6}} \, V_2 \, \ell_p^9 \, U_H^2 = \frac{512\pi^9}{27}\sqrt{2N_2^3 N_6} \, V_2 \, \ell_p^9 \, T_H^2.$$

(4.8)

The horizon area is related to the entropy of the black brane via the Bekenstein-Hawking relation

$$S = \frac{A}{4G_{11}} = \frac{8\pi^2}{27}\sqrt{2N_2^3 N_6} \, V_2 \, T_H^2,$$

(4.9)

[4]The N_2 and N_6 dependence of our result differs from that in eq. (3.23) in ref. [11].

where $G_{11} = 2^4 \pi^7 \ell_p^9$. On the other hand, the breakdown of our approximation at the horizon may signal that higher-order effects are important, which may alter these conclusions. We have not been able to compute the $1/m^4$ corrections to the supergravity solution.

As a check on (4.9), we may compute the entropy from the first law of thermodynamics, following the method described in refs. [14, 3]. The four-dimensional part of the decoupled metric (4.3) has the form of a black hole in AdS_4 with radius $\frac{1}{2} \left(32\pi^2 N_2 N_6 \right)^{1/6} \ell_p$. The Euclidean action for the black hole is given by

$$I = -\frac{1}{16\pi G_4} \int \mathrm{d}^4 x \sqrt{-g_4} \left(R_4 - 2\Lambda \right) , \tag{4.10}$$

where $R_4 = -48/\left(32\pi^2 N_2 N_6 \right)^{1/3} \ell_p^2$ and $\Lambda = -12/\left(32\pi^2 N_2 N_6 \right)^{1/3} \ell_p^2$. Evaluating the action using $1/G_4 = \mathrm{Vol}_7/G_{11}$, where $G_{11} = 2^4 \pi^7 \ell_p^9$ and $\mathrm{Vol}_7 = \left(\pi^4/3 N_6 \right) \left(32\pi^2 N_2 N_6 \right)^{7/6} \ell_p^7$ we obtain

$$I = \frac{V_2}{64\pi^4 N_6} \int_{U_H}^U U^2 \, \mathrm{d}U \int_0^\beta \mathrm{d}\tau , \tag{4.11}$$

where we have cut off the divergent integral at some large value U, and where the period of the Euclidean time τ for the non-extremal metric is given by

$$\beta = \frac{1}{T_H} = \frac{8\pi^2 \sqrt{2 N_2 N_6}}{3 U_H} . \tag{4.12}$$

To obtain a finite result, one must subtract the action for the extremal metric

$$I_{\mathrm{e}} = \frac{V_2}{64\pi^4 N_6} \int_0^U U^2 \, \mathrm{d}U \int_0^{\beta'} \mathrm{d}\tau . \tag{4.13}$$

Following refs. [14, 3], one must adjust the period β' of the extremal metric so that the geometry is the same for extremal and nonextremal metrics at the hypersurface at U. This implies

$$\beta' = \beta \left[1 - \left(\frac{U_H}{U} \right)^3 \right]^{1/2} . \tag{4.14}$$

The regularized action is

$$I - I_{\mathrm{e}} = \frac{V_2}{64\pi^4 N_6} \left[\int_{U_H}^U U^2 \, \mathrm{d}U \int_0^\beta \mathrm{d}\tau - \int_0^U U^2 \, \mathrm{d}U \int_0^{\beta'} \mathrm{d}\tau \right] , \tag{4.15}$$

which, in the $U \to \infty$ limit, gives

$$\Delta I = \lim_{U \to \infty} \left(I - I_{\mathrm{e}} \right) = -\frac{V_2}{384\pi^4 N_6} U_H^3 \beta = -\frac{8\pi^2 V_2}{81} \sqrt{2 N_2^3 N_6} \beta^{-2} \tag{4.16}$$

The energy is $E = \frac{\partial}{\partial \beta} \Delta I$, so that the entropy, as given by the first law of thermodynamics, is

$$S = \beta E - \Delta I = \frac{8\pi^2}{27} \sqrt{2 N_2^3 N_6} \, V_2 \, T_H^2 , \tag{4.17}$$

which agrees with the entropy (4.9) calculated using the Bekenstein-Hawking relation. This computation is valid through order $1/m^2$.

5   Reduction to ten dimensions

The ψ direction is an isometry of the eleven-dimensional metric (3.10), and so we can obtain the 10-dimensional metric from the 11-dimensional one by dimensional reduction along the ψ direction.[5] The 10-dimensional metric in the string frame $\mathrm{d}s_{\mathrm{str}}^2$ and the dilaton ϕ are identified through the relation

$$\mathrm{d}s_{11}^2 = \mathrm{e}^{4(\phi-\phi_\infty)/3}\left[R_\# \, \mathrm{d}\psi + A_\mu \, \mathrm{d}x^\mu\right]^2 + \mathrm{e}^{-2(\phi-\phi_\infty)/3}\mathrm{d}s_{\mathrm{str}}^2 \tag{5.1}$$

where $R_\#$ is the asymptotic radius of the eleventh dimension

$$R_\# = 4m = g_s\ell_s, \qquad g_s = \mathrm{e}^{\phi_\infty} . \tag{5.2}$$

Comparing (5.1) with (3.10), one obtains the dilaton

$$\mathrm{e}^\phi = g_s H^{1/4}\left(\frac{B}{m}\right)^{3/2}\left[1 - \frac{B^2}{m^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6)\right] \tag{5.3}$$

and the 10-dimensional metric in the string frame

$$
\begin{aligned}
\mathrm{d}s_{\mathrm{str}}^2 \;=\;& H^{-1/2}\frac{B}{m}\left[1 - \frac{2B^2}{3m^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6)\right]\left(-f_1\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) \\
+\;& H^{1/2}\frac{B}{m}\left[1 - \frac{2B^2}{3m^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}; -\tfrac{2}{3}; R_H^6/R^6)\right]\left(f_1^{-1}\mathrm{d}R^2 + R^2\mathrm{d}\beta^2 + R^2\cos^2\beta\mathrm{d}\Omega_3^2\right) \\
+\;& H^{1/2}\frac{B}{m}\left(\frac{R\sin\beta}{2}\right)^2\left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right) + \mathcal{O}\left(\frac{1}{m^4}\right) .
\end{aligned}
\tag{5.4}
$$

If we now take the decoupling limit, the dilaton becomes

$$\mathrm{e}^\phi = \left(\frac{32\pi^2 N_2 \sin^6\beta}{N_6^5}\right)^{1/4}\left[1 - \frac{U\sin^2\beta}{g_{\mathrm{YM}}^2 N_6^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}; U_H^3/U^3)\right] + \mathcal{O}\left(\frac{1}{m^4}\right) \tag{5.5}$$

and the 10-dimensional string metric becomes

$$
\begin{aligned}
\mathrm{d}s_{\mathrm{str}}^2 =\;& \frac{\ell_s^2\sin\beta}{N_6}\left\{\frac{U^2}{(32\pi^2 N_2 N_6)^{1/2}}\left[1 - \frac{2U\sin^2\beta}{3g_{\mathrm{YM}}^2 N_6^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}; U_H^3/U^3)\right]\left(-f_1\mathrm{d}t^2 + \right. \right. \\
& \left. \mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + (32\pi^2 N_2 N_6)^{1/2}\left[1 - \frac{2U\sin^2\beta}{3g_{\mathrm{YM}}^2 N_6^2} \, {}_2F_1(-\tfrac{1}{3}, -\tfrac{1}{3}, -\tfrac{2}{3}; U_H^3/U^3)\right]\left[f_1^{-1}\frac{\mathrm{d}U^2}{4U^2} + \right. \\
& \left. \left. \mathrm{d}\beta^2 + \cos^2\beta \, \mathrm{d}\Omega_3^2\right] + \tfrac{1}{4}(32\pi^2 N_2 N_6)^{1/2}\sin^2\beta\left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)\right\} + \mathcal{O}\left(\frac{1}{m^4}\right) .
\end{aligned}
\tag{5.6}
$$

In the $m \to \infty$ limit, the dilaton will be small provided

$$\frac{32\pi^2 N_2 \sin^6\beta}{N_6^5} \ll 1 \tag{5.7}$$

---

[5]In the absence of D6 branes, i.e., in a flat 11-dimensional background, the M2 supergravity solution must be smeared in the ψ direction before it can be reduced. In the presence of D6 branes, the ψ direction is "pinched off" at the D6 (and therefore D2) brane location, so no smearing is necessary.

as discussed in ref. [11], and the reduction to ten dimensions is only valid in that case.

Also, as discussed in the previous section, the $1/m$ expansion will be valid only if

$$U_H \ll U \ll \frac{g_{\rm YM}^2 N_6^2}{\sin^2 \beta} \,. \tag{5.8}$$

Since (unlike the Hawking temperature and horizon area) the dilaton depends explicitly on f_2, whose $1/m^2$ contribution diverges (logarithmically) at the horizon, the reduction to 10-dimensional supergravity seems to be problematic, at least near the horizon, although it seems legitimate away from the horizon. (As mentioned above, f_2 summed to all orders in $1/m$ might indeed be regular at the horizon, but we have no way of knowing whether the dilaton remains small there.)

The ten-dimensional Ricci curvature computed from the decoupled metric (5.6) is

$$R_{10} = \frac{3}{8\pi} \sqrt{\frac{N_6}{2N_2}} \frac{(1 - 15 \cos 2\beta)}{\sin^3 \beta} \frac{1}{\ell_s^2} + \mathcal{O}\left(\frac{1}{m^2}\right) \tag{5.9}$$

hence for $N_6 \gg N_2$, the ten-dimensional supergravity solution is no longer applicable, and the theory is described by a weakly-coupled field theory[11].

An estimate of the transition from the supergravity description to that of the weakly-coupled gauge theory is $N_2 \sim N_6 = N$, which can be considered as a correspondence point. At this correspondence point, the black brane entropy (4.17) goes as $S \sim N^2 T^2$, consistent with the behavior expected for the high-temperature deconfining phase of the gauge theory (1.3).

Thus, we see that the non-extremal localized D2–D6 brane system presents the expected high-temperature thermodynamics, which is compatible with evolution from strong to weak 't Hooft coupling, *i.e.*, from small to large curvature in the geometry. We also note that the calculation presented here gives just the high-temperature limit of the system. Therefore, we have no opportunity to observe a possible Hawking–Page transition in the bulk [14, 3], or the 3$^{\rm rd}$ order phase-transition found in the weakly-coupled gauge theory [6].

6 Discussion

In this paper we have obtained a non-extremal version of the metric that describes the localized intersection of D2 and D6 branes given by Cherkis and Hashimoto in ref. [8]. The non-extremal version of this metric was found as a systematic expansion in the neighborhood of the core of the D6 branes (an expansion in $1/m$, where the parameter m gives the radius of the Taub-NUT space), but to all orders in the non-extremal parameters. Certain restrictions were discussed which must be respected for the geometric description and $1/m$ expansion to be valid. It was found that this expansion breaks down near the horizon due to a logarithmic divergence of the hypergeometric function appearing in the non-extremal metric. Nevertheless, proceeding formally, we found that the Hawking temperature and Bekenstein-Hawking entropy do not depend on the hypergeometric function, so that in fact these thermodynamic quantities are uncorrected by terms of $\mathcal{O}(1/m^2)$. One could argue thus that (4.7) and (4.9) will survive even with a better understanding of the non-extremal metric, as evidenced by the

computation of the entropy from the action in (4.17) and by the fact that (4.9) is compatible with the field theory result (1.3) at the correspondence point $N_2 \sim N_6 = N$. Also note that the non-extremal metric we have found corresponds to the high temperature limit, so that one is unable to consider a possible Hawking-Page transition in the bulk, which may be dual to the Gross-Witten phase transition [16] found in the weakly-coupled gauge theories [6].

Of concern is that our systematic expansion breaks down near the horizon, as we already mentioned in the previous paragraph. We can envision at least two possibilities. The first is that the result is actually finite at the horizon when all orders in $1/m$ are included. The second is that this divergence is genuine, but may be cut-off by separating the D2 from the D6 branes, giving a small mass to the fields in the fundamental representation of the gauge group (the separation between D2 and D6 branes was considered in [8] for the extremal case). One might then consider the logarithmic divergence (when the mass of the fundamentals vanishes) as a delocalization effect analogous to that discussed by [15]. These are issues for future study.

Another possible extension of this work would be to study the interpolation between strong and weak coupling along the lines of ref. [1], in which the free energy for the four-dimensional $\mathcal{N} = 4$ SU(N) gauge theory is studied. In the large N limit, the entropy is $N^2 T^3$ times a function of the 't Hooft coupling $f(g_{\mathrm{YM}}^2 N)$. In the strong-coupling limit, the entropy was found to be 3/4 that of the weak-coupling limit, with corrections of order $(g_{\mathrm{YM}}^2 N)^{-3/2}$, coming from R^4 terms in the IIB effective string action. In our case, we anticipate an analogous, but richer situation, where we expect our strong-coupling result (1.4) to be corrected by a function $f(g_{\mathrm{YM}}^2 N_2/U, \ N_6/N_2)$, interpolating between strong (IR) and weak (UV) domains.

Acknowledgments

References

[1] S. S. Gubser, I. R. Klebanov and A. W. Peet, "Entropy and Temperature of Black 3-Branes," Phys. Rev. D **54**, 3915 (1996) [arXiv:hep-th/9602135]. I. R. Klebanov and A. A. Tseytlin, "Entropy of Near-Extremal Black p-branes," Nucl. Phys. B **475**, 164 (1996) [arXiv:hep-th/9604089]. S. S. Gubser, I. R. Klebanov and A. A. Tseytlin, "Coupling constant dependence in the thermodynamics of N = 4 supersymmetric Yang-Mills theory," Nucl. Phys. B **534**, 202 (1998) [arXiv:hep-th/9805156].

[2] A. Fotopoulos and T. R. Taylor, "Comment on two-loop free energy in N = 4 supersymmetric Yang-Mills theory at finite temperature," Phys. Rev. D **59**, 061701 (1999) [arXiv:hep-th/9811224].

[3] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[4] B. Sundborg, "The Hagedorn transition, deconfinement and N = 4 SYM theory," Nucl. Phys. B **573**, 349 (2000) [arXiv:hep-th/9908001].

[5] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk, "The Hagedorn/deconfinement phase transition in weakly-coupled large N gauge theories," arXiv:hep-th/0310285. H. Liu, "Fine structure of Hagedorn transitions," arXiv:hep-th/0408001.

[6] H. J. Schnitzer, "Confinement/deconfinement transition of large N gauge theories with N_f fundamentals: N_f/N finite," Nucl. Phys. B **695**, 267 (2004) [arXiv:hep-th/0402219].

[7] B. S. Skagerstam, "On The Large N_c Limit Of The SU(N_c) Color Quark - Gluon Partition Function," Z. Phys. C **24**, 97 (1984). A. Dumitru, J. Lenaghan and R. D. Pisarski, "Deconfinement in matrix models about the Gross-Witten point," arXiv:hep-ph/0410294.

[8] S. A. Cherkis and A. Hashimoto, "Supergravity solution of intersecting branes and AdS/CFT with flavor," JHEP **0211**, 036 (2002) [arXiv:hep-th/0210105].

[9] N. Itzhaki, A. A. Tseytlin and S. Yankielowicz, "Supergravity solutions for branes localized within branes," Phys. Lett. B **432**, 298 (1998) [arXiv:hep-th/9803103].

[10] J. Erdmenger and I. Kirsch, "Mesons in gauge/gravity dual with large number of fundamental fields," arXiv:hep-th/0408113.

[11] O. Pelc and R. Siebelink, "The D2-D6 system and a fibered AdS geometry," Nucl. Phys. B **558**, 127 (1999) [arXiv:hep-th/9902045].

[12] M. Cvetic and A. A. Tseytlin, "Non-extreme black holes from non-extreme intersecting M-branes," Nucl. Phys. B **478**, 181 (1996) [arXiv:hep-th/9606033].

[13] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[14] S. W. Hawking and D. N. Page, "Thermodynamics Of Black Holes In Anti-De Sitter Space," Commun. Math. Phys. **87**, 577 (1983).

[15] D. Marolf and A. W. Peet, "Brane baldness vs. superselection sectors," Phys. Rev. D **60**, 105007 (1999) [arXiv:hep-th/9903213].

[16] D. J. Gross and E. Witten, "Possible Third Order Phase Transition In The Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446 (1980).